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Application of mixed Quadrature rules using anti-Gaussian Quadrature rule in Adaptive quadrature

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Abstract A model is set up which embodies the basic features of Adaptive quadrature routines involving mixed rules. Not before mixed quadrature rules basing on anti-Gaussian quadrature rule have been used for fixing termination criterion in Adaptive quadrature routines. Two mixed quadrature rules of higher precision for approximate evaluation of real definite integrals have been constructed using an anti-Gaussian rule for this purpose. The first is the linear combination of anti-Gaussian three point rule and Simpsons $1/3^{rd}$ rule , the second is the linear combination of anti-Gaussian three point rule and Simpsons $3/8^{th}$ rule. The analytical convergence of the rules have been studied. The error bounds have been determined asymptotically. Adaptive quadrature routines being recursive by nature, a termination criterion is formed taking in to account two mixed quadrature rules. The algorithm presented in this paper has been "C" programmed and successfully tested on different integrals. The efficiency of the process is reflected in the table at the end.

Keywords and phrases Gauss Legendre two point rule, anti-Gaussian rule, Simpsons $1/3^{rd}$ rule, Simpsons $3/8^{th}$ rule, mixed quadrature rule , Adaptive quadrature.

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1. Introduction

Given a real integrable function f an interval $[a, b]$ and a prescribed tolerance ε, it is desired to compute an approximation P to the integral$I = \int_a^b f(x)dx$, so that $|P - I| \leq \varepsilon$. This can be done following adaptive integration schemes developed in papers [2,3,6,7,8,9]. In adaptive integration, the points at which the integrand is evaluated ,are chosen in a way that depends on the nature of the integrand. The basic principle of adaptive quadrature routines is discussed in the following manner.

If c is any point between a and b ,then

$$\int_a^b f\left(x\right)dx = \int_a^c f\left(x\right)dx + \int_c^b f\left(x\right)dx$$

The idea is that if we can approximate each of the two integrals on the right to within a specified tolerance, then the sum gives us the desired result. If not we can recursively apply the adaptive property to each of the intervals $[a, c]$ and $[c, b]$. Adaptive subdivision of course has geometrical appeal. It seems intuitive that points should be concentrated in regions where the integrand is badly behaved. The whole interval rules can take no direct account of this.

In this paper we design an algorithm for numerical computation of integrals in the adaptive quadrature routines involving mixed rules. The literature of the mixed quadrature rule [5,6,7,8,9] involves construction of a symmetric quadrature rule of higher precision as a linear/convex combination of two other rules of equal lower precision.

So far as anti-Gaussian quadrature is concerned, Dirk P. Laurie [1] is the first person to coin the idea of anti-Gaussian quadrature formula . An anti-Gaussian quadrature formula is an $(n+1)$ point formula of degree $(2n-1)$ which integrates all polynominals of degree up to $(2n+1)$ with an error equal in magnitude but opposite in sign to that of n-point Gaussian formula.

If $H^{(n+1)}(f)= \sum_{i=1}^{n+1} \lambda_i f(\xi_i)$ be $(n+1)$ point anti-Gaussian formula and $G^{(n)}(p)$ be n point Gaussian formula, then by hypothesis :

$I(p) - H^{(n+1)}(p)$ = - $(I(p) - G^{(n)}(p)), p \in P_{2n+1}$.Where p is a polynomial of degree $\leq (2n+1)$. In this paper we design a three point anti-Gaussian rule following LAURIE[1].

As the anti-Gaussian three point rule $RH_w^3(f)$ [1] and Simpsons $1/3^{rd}$ rule $RSm1/3(f)$ rules are of same precision (i.e precision 3), one can form a mixed quadrature rule $RH_w^3Sm1/3(f)$ of precision five by taking a suitable linear combination of these two rules. Similarly one can form a mixed quadrature rule $RH_w^3Sm3/8(f)$of precision five by taking linear combination of the anti-Gaussian three point rule $RH_w^3(f)$ and Simpsons3/8th rule $RSm3/8(f)$.

After Laurie, use of anti-Gaussian quadrature is not seen in literature.In this paper, first time we incorporate the idea of anti-Gaussian quadrature to form mixed quadrature rules. Further, we use this type of mixed rules in adaptive quadrature routines.

To prepare an algorithm for adaptive quadrature routines in evaluating an integral $I = \int_a^b f(x)dx$,we use the following two mixed quadrature rules.

(i) $RH_w^3Sm1/3(f)$ as I_1

(ii) $RH_w^3Sm3/8(f)$ as I_2.

The adaptive strategies can be found in standard texts in numerical analysis.

2. Construction of anti-Gaussian three point rule from Gauss-legendre two point rule.

We choose the Gauss-Legendre two point rule :

$$G_w^2(f) = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) \tag{1}$$

We develop a three point anti-Gaussian rule $RH_w^3(f)$from two point Gaussian rule $G_w^2(f)$ following Laurie.

We take $RH_w^3(f) = \alpha_1 f(\xi_1) + \alpha_2 f(\xi_2) + \alpha_3 f(\xi_3)$.

In order to obtain the unknown weights and nodes, we assume that

(i) The rule is exact for all polynomial of degree ≤ 3.

(ii) The rule integrates all polynomials of degree up to five with an error equal in magnitude and opposite in sign to that of Gaussian rule. Thus we obtain a system of six equations having six unknowns namely α_i,ξ_i $(i = 1, 2, 3)$.

Solving the resulting system of equations we get,

$$\alpha_1 = \frac{5}{13} = \alpha_3, \alpha_2 = \frac{16}{13}, \xi_1 = \pm\sqrt{\frac{13}{15}}, \xi_2 = 0, \xi_3 = \mp\sqrt{\frac{13}{15}}$$

Hence, the method becomes

$$\int_{-1}^{1} f(x)dx \approx RH_w^3(f) = \frac{5}{13} f\left(-\sqrt{\frac{13}{15}}\right) + \frac{16}{13} f(0) + \frac{5}{13} f\left(\sqrt{\frac{13}{15}}\right). \tag{2}$$

The error associated with the method (2) is computed as

$$EH_w^3(f) = \int_{-1}^{1} f(x)dx - RH_w^3(f) = \frac{-f^{iv}(0)}{135} - \frac{1016 f^{vi}(0)}{7! \times 675} + \ldots \tag{3}$$

3. Construction of mixed quadrature rule by using anti-Gaussian three point rule with Simpson's 1/3rd rule

We have the anti-Gaussian three point rule:

$$RH_w^3(f) = \frac{5}{13} f\left(-\sqrt{\frac{13}{15}}\right) + \frac{16}{13} f\left(0\right) + \frac{5}{13} f\left(\sqrt{\frac{13}{15}}\right) \tag{4}$$

and Simpson's $1/3^{rd}$ rule

$$RSm1/3(f) = \frac{1}{3}\left[f\left(-1\right) + 4f\left(0\right) + f\left(1\right)\right] \tag{5}$$

Each of the rules $RH_w^3(f)$ and $RSm1/3(f)$ is of precision three.

Let $EH_w^3(f)$ and $ESm1/3(f)$ denote the errors in approximating the integral $I(f)$by the rules $RH_w^3(f)$ and $RSm1/3(f)$ respectively.

Now

$$I(f) = RH_w^3(f) + EH_w^3(f) \tag{6}$$

$$I(f) = RSm1/3(f) + ESm1/3(f) \tag{7}$$

Using Maclaurines expansion of the functions in equation (4) and (5),we have

$$EH_w^3(f) = \frac{-f^{iv}(0)}{135} - \frac{2 \times 508}{7! \times 675} f^{vi}(0) + \ldots \tag{8}$$

$$ESm1/3(f) = \frac{-1}{90} f^{iv}(0) - \frac{8}{7! \times 675} f^{vi}(0) + \ldots \tag{9}$$

Eliminating $f^{iv}(0)$ from equation (8) and (9) we have

$$I(f) = 3RH_w^3(f) - 2RSm1/3(f) + 3EH_W^3(f) - 2ESm1/3(f). \tag{10}$$

or

$$I(f) = RH_w^3Sm1/3(f) + EH_w^3Sm1/3(f) \tag{11}$$

where

$$RH_w^3Sm1/3(f) = 3RH_w^3(f) - 2RSm1/3(f) \tag{12}$$

$$RH_w^3Sm1/3(f) = \frac{15}{13}f\left(-\sqrt{\frac{13}{15}}\right) + \frac{48}{13}f(0)$$

$$+\frac{15}{13}f\left(\sqrt{\frac{13}{15}}\right) - \frac{2}{3}[f(-1) + 4f(0) + f(1)] \tag{13}$$

This is the desired mixed Quadrature rule of precision five for the approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by.

$$EH_w^3Sm1/3(f) = 3EH_w^3(f) - 2ESm1/3(f) \tag{14}$$

or

$$EH_w^3Sm1/3(f) = \frac{184}{7! \times 225}f^{vi}(0) + ... \tag{15}$$

$$\left|EH_w^3Sm1/3(f)\right| \cong \frac{184}{7! \times 225}\left|f^{vi}(\xi)\right|, \xi \in [-1, 1] \tag{16}$$

The rule $RH_w^3Sm1/3(f)$ is called a mixed type rule of precision five as it is constructed from two different types of the rules of the same precision .

4. Construction of mixed Quadrature rule by using anti-Gaussian three point rule with Simpsons $3/8^{th}$ rule:

Anti-Gaussian three point rule $RH_w^3(f)$ and Simpson's $3/8^{th}$ rule$RSm3/8(f)$ are mixed in the same manner as described in article 3 to get another mixed rule $RH_w^3Sm3/8(f)$ and the corresponding error $EH_w^3Sm3/8(f)$.

Where

$$RH_w^3Sm3/8(f) = \frac{3}{4}\left[f(-1) + 3\left\{f\left(-\frac{1}{3}\right) + f\left(\frac{1}{3}\right)\right\} + f(1)\right]$$

$$-\left[\frac{10}{13}\left\{f\left(-\sqrt{\frac{13}{15}}\right) + f\left(\sqrt{\frac{13}{15}}\right)\right\} + \frac{32}{13}f(0)\right] \tag{17}$$

This mixed quadrature rule is also of precision five.

And

$$EH_w^3Sm3/8(f) = \left(\frac{23}{135 \times 729 \times 105} - \frac{127x2}{405 \times 630 \times 675} \right) f^{vi}(0) + \ldots \quad (18)$$

$$\mid EH_w^3Sm3/8(f) \mid \leq \mid \left(\frac{23}{135 \times 729 \times 105} - \frac{127x2}{405 \times 630 \times 675} \right) f^{vi}(\xi) \mid , \xi \in [-1, 1]$$

5. Error analysis

An asymptotic error estimate and an error bound of the rule (12) are given below.

Theorem 1. *Let $f(x)$ be sufficiently differentiable function in the closed interval $[-1, 1]$. Then the error $EH_w^3Sm1/3(f)$ associated with the rule $RH_w^3Sm1/3(f)$ is given by*

$$\left| EH_w^3Sm1/3(f) \right| \leq \frac{184}{7! \times 225} \left| f^{vi}(\xi) \right| , \xi \in [-1, 1] \quad (19)$$

Proof : From (12) and (14) we have

$$RH_w^3Sm1/3(f) = 3RH_w^3(f) - 2RSm1/3(f)$$

And the truncation error generated in this approximation is given by

$$EH_w^3Sm1/3(f) = 3EH_w^3(f) - 2ESm1/3(f) = \frac{184}{7! \times 225} f^{vi}(0) + \ldots$$

Hence we have $\left| EH_w^3Sm1/3(f) \right| \approx \frac{184}{7! \times 225} \left| f^{vi}(0) \right|$.

Theorem 2. *Let $f(x)$ be sufficiently differentiable function in the closed interval*

Let $f(x)$ be sufficiently differentiable function in the closed interval $[-1, 1]$ .Then the error

$EH_w^3Sm3/8(f)$ associated with the rule $RH_w^3Sm3/8(f)$ is given by

$$\left| EH_w^3Sm3/8(f) \right| \leq \frac{15714}{315 \times 675 \times 243} \mid f^{vi}(\xi) \mid , \xi \in [-1, 1] \quad (20)$$

Proof : Similar to Theorem 5.1.

6. Numerical verification

Table 1. Comparison among the rules $RGL_2(f)$, $RH_w^3(f)$, $RSm1/3(f)$, $RSm3/8(f)$, $RH_w^3Sm1/3(f)$, $RH_w^3Sm3/8(f)$, for approximation of the integrals in the whole interval method

Integrals	Exact value(I)	Approximate Value					
		$RGL_2(f)$	$RH_w^3(f)$	$RSm1/3(f)$	$RSm3/8(f)$	$RH_w^3Sm1/3(f)$	$RH_w^3Sm3/8(f)$
$I = \int_0^1 e^{-x^2}dx$	0.746825	0.746594	0.747054	0.747180	0.79699231	0.7468012	0.74686889
$I_6=\int_0^1 \sqrt{x}\sin x dx$	0.3642219	0.3632212	0.3652365	0.3662485	0.36535991	0.36321199	0.36560703

Table 2. Approximation of the integrals in the adaptive quadrature routines:

Integrals	Exact Value(I)	Approximate Value $RGL_2(f)$	No of step	Error	Approximate Value $RH_w^3(f)$	No of step	Error	Prescribed Tolerance
$I = \int_0^1 e^{-x^2}dx$	0.746825	0.74652412	15	0.0000008	0.746824138	15	0.00000086	0.00001
$I_6=\int_0^1 \sqrt{x}\sin x dx$	0.3642219	0.36422157	11	0.0000003	0.3642222	11	0.0000003	0.00001

Table 3. Approximation of the integrals in the adaptive quadrature routines:

Integrals	Exact Value(I)	Approximate Value $RSim1/3(f)$	No of step	Error	Approximate Value $RSim3/8(f)$	No of step	Error	Prescribed Tolerance
$I = \int_0^1 e^{-x^2}dx$	0.746825	0.74682414	15	0.0000008	0.74682418	08	0.0000008	0.00001
$I_6=\int_0^1 \sqrt{x}\sin x dx$	0.3642219	0.3642225	11	0.000192	0.36422257	09	0.00000067	0.00001

Table 4. Approximation of the integrals in the adaptive quadrature routines:

Integrals	Exact Value(I)	Approximate Value $RH_w^3Sim1/3(f)$	No of step	Error	Approximate Value $RH_w^3Sim3/8(f)$	No of step	Error	Prescribed Tolerance
$I = \int_0^1 e^{-x^2}dx$	0.746825	0.74682413	03	0.0000008	0.74682413	03	0.0000008	0.00001
$I_6=\int_0^1 \sqrt{x}\sin x dx$	0.3642219	0.3642209	07	0.0000009	0.36422218	09	0.00000028	0.00001

7. Observation

From the table (1) it is observed that the results obtained due to the mixed rules$RH_w^3Sm1/3(f)$ and $RH_w^3Sm3/8(f)$ are better than their constituent rules $RGL_2, RH_w^3(f), RSm1/3(f), RSm3/8(f)$ when applied on whole interval. However when these rules are used in adaptive mode, tables (2),(3),(4) depict that the mixed quadrature rules $RH_w^3Sm1/3(f)$ and $RH_w^3Sm3/8(f)$,give very good result and take less number of steps than its constituent rules. Even the results are better than the results of previously solved papers [6,7,8].

8. Conclusion

In this paper ,we have concentrated mainly on computation of definite integrals in the adaptive quadrature routines involving mixed quadrature rules. We observed that mixed quadrature rules formed using anti-Gaussian quadrature can very well be used for evaluating real definite integrals in the adaptive quadrature routines.

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